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On the Stability and Hopf Bifurcation of a Shimizu Morioka Model

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Abstract

This paper presents a three-dimensional continuous autonomous chaotic system is called Shimizu Morioka system (SMS) with four terms and two quadratic nonlinearities. The new system contains five variational parameters and exhibits Lorenz and Rossler like attractors in numerical simulations. The basic dynamical properties of the new system are analyzed by means of equilibrium points, eigenvalue structures. Some of the basic dynamic behavior of the system is explored further investigation in the Lyapunov exponent, and we show that Shimizu Morioka system is almost linear system. Finally, a numerical example is given to support the analytic results.

Keywords: Shimizu Morioka system; Stability; Hopf bifurcation; Almost linear system.

1. Introduction

The science of nonlinear dynamics and chaos theory has sparked many researchers to develop mathematical models that simulate vector fields of nonlinear chaotic physical systems. Nonlinear phenomena arise in all fields of engineering, physics, chemistry, biology, economics, and sociology. Examples of nonlinear chaotic systems include planetary climate prediction models, neural network models, data compression, turbulence, nonlinear dynamical economics, information processing, preventing the collapse of power systems, high-performance circuits and devices, and liquid mixing with low power consumption [1]. The Lorenz system of differential equations arose from the work of meteorologist mathematician Edward N. Lorenz, who was studying thermal variations in an air cell underneath a thunderhead [2-4].

The defining equation of the Shimizu Morioka system are:

$$\left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= x-\alpha y - xz\\ \dot{z} &= x^2-\beta z\end{aligned}\right\} \qquad (1)$$

where $(x,y,z)\in R^3$ and α,β with $\alpha\neq 0$ are the real parameters . The Shimizu Morioka system is derived from the Lorenz system [5]. It help us to understand better the family of Lorenz systems. As far as we know, not many systems derived from Lorenz was studied. Originally, the Lorenz system was a model of convection. Convection is close related to the dynamics of the atmosphere. Warm air with a lower density rises to higher altitudes where it becomes cool and its density increases. Falling again it creates the atmospheric currents. It was modeled in laboratory and it is known as Rayleigh-Benard experiment. Good notes on Lorenz system can be found in [11].

In the last decades the nonlinear system was intensively studied because the nonlinear phenomena are met in many areas, from engineering to human brain and heart disease [7,8]. not many systems derived from the Lorenz system have been studied. We record here two of them: Lü system [9] and Chen system [13]. Good notes on these systems are reported in [10,14-17].The word chaos is familiar in every day speech. It normally means at lack of order or predictability. Thus one says that the weather is chaotic, or that rising particles of smoke are chaotic, or that the stock market is chaotic. Both sensitive dependence on initial conditions and Lyapunov exponent qualify as measure of unpredictability [18]. The aim of this paper is to study the stability and a Hopf bifurcation from equilibrium by taking one coefficient as bifurcation parameter, and show that the system is almost linear system. Finally, a numerical example is given to support the analytic results.

2. Stability analysis of the Shimizu Morioka system

Proposition 1. If $\beta > 0$, then the Shimizu Morioka system has three isolated equlibria $O(0,0,0)$, $E_1(\sqrt{\beta}, 0,1), E_2(-\sqrt{\beta},0,1)$ and for $\beta \le 0$ it has only one isolated equilibrium $O(0,0,0)$.

Proof: Solving the system

$$y = 0$$
$$x - \alpha y - xz = 0 \;\Rightarrow\; x(1-z) = 0 \;\Rightarrow\; x = 0, z = 1$$
$$x^2 - \beta z = 0 \;\Rightarrow\; x = \pm\sqrt{\beta z} \;\Rightarrow\; x = \pm\sqrt{\beta}$$

which yields $x = 0, y = 0, z = 0$ and for $\beta > 0$, $x = \pm\sqrt{\beta}$, $y = 0$, $z = 1$.

Therefore, the system (1) has only one equilibrium $O(0,0,0)$ for $\beta \le 0$ but has three isolated equlibria:

$O(0,0,0)$, $E_1(\sqrt{\beta}, 0,1), E_2(-\sqrt{\beta},0,1)$ for $\beta > 0$.

Definition 1. [12]

The critical point $\overline{X}$ of the nonlinear vector field $\dot{x} = f(x)$, $x \in R^n$ is asymptotically stable if that all of the eigenvalues of Jacobian matrix $Df(\overline{x})$ have negative real parts.

Remark 1.[12]

The critical point $\overline{x}$ is said to be unstable if at least one eigenvalues of $Df(\overline{x})$ has positive real part.

Theorem 1.

The point $O(0,0,0)$ is unstable.

Proof: The Jacobian matrix of system (1) at the point $O(0,0,0)$ is:

$$J(O) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\alpha & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

The characteristic polynomial of $J(O)$ is :.

$$\Rightarrow (-\beta - \lambda)(\lambda^2 + \alpha\lambda - 1) = 0.$$

Then, the eigenvalues of $J(O)$ are: $\lambda_1 = -\beta$, $\lambda_2 = \frac{1}{2}(-\alpha - \sqrt{\alpha^2 + 4}), \lambda_3 = \frac{1}{2}(-\alpha + \sqrt{\alpha^2 + 4})$.

If $\beta < 0$ then $\lambda_1 > 0$. If $\alpha < 0$ then $\lambda_3 > 0$, and if $\alpha > 0 \Rightarrow \lambda_3 > 0$. Consequently, the point $O(0,0,0)$ is an unstable.

Next, consider the stability of system (1) at

$E_1(\sqrt{\beta}, 0,1), E_2(-\sqrt{\beta},0,1)$ for $\beta > 0$. Because the system is invariant under the transformation $(x,y,z) \to (-x,y,z)$, one only needs to consider the stability of system (1) at $E_1(\sqrt{\beta}, 0,1)$.

Theorem 2.

The equilibrium point $E_1(\sqrt{\beta}, 0, 1)$ is asymptotically stable if and only if $\alpha+\beta>0$, $\beta>0$ and $\beta(\alpha^2+\alpha\beta-2)>0$.

Proof: Let

$$x = X+\sqrt{\beta}$$
$$y = Y$$
$$z = Z+1$$

The system (1) becomes:

$$\left.\begin{aligned}\dot{X} &= Y\\ \dot{Y} &= -XZ-\alpha Y-\sqrt{\beta}\,Z\\ \dot{Z} &= X^2-\beta Z\end{aligned}\right\} \tag{2}$$

Hence, one has to consider the stability of system (2) at $(0,0,0)$.

The Jacobian matrix of system (2) at the point $(0,0,0)$ is:

$$J=\begin{pmatrix} 0 & 1 & 0\\ 0 & -\alpha & -\sqrt{\beta}\\ 2\sqrt{\beta} & 0 & -\beta\end{pmatrix}$$

The characteristic polynomial of J is :

$$\lambda^3+(\alpha+\beta)\lambda^2+\alpha\beta\lambda+2\beta=0. \tag{3}$$

Let $A=\alpha+\beta$, $B=\alpha\beta$, $C=2\beta$.

By Routh-Hurwits conditions, the roots equation (3) have strictly negative real parts if and only if $A>0, C>0$ and $AB-C>0$. Then we have:

$$\left.\begin{aligned}&\alpha+\beta>0\\ &\beta>0\\ &\beta(\alpha^2+\alpha\beta-2)>0\end{aligned}\right\} \tag{4}$$

3. Hopf bifurcation of the Shimizu Morioka System.

Consider the parameter β as bifurcation parameter.

(i)Bifurcations at the point $O(0,0,0)$

Proposition 2. The equilibrium point $O(0,0,0)$ cannot undergo a Hopf bifurcation.

Proof: the roots of the characteristics polynomials of the Jacobian matrix of the system (1) at the point $O(0,0,0)$ are:

$\lambda_1=-\beta$, $\lambda_2=\frac{1}{2}(-\alpha-\sqrt{\alpha^2+4})$, $\lambda_3=\frac{1}{2}(-\alpha+\sqrt{\alpha^2+4})$ and the last two roots cannot be purely imaginary for all $\alpha\neq 0$.

Hence, the Hopf bifurcation is not exist.

(ii) Bifurcations of the points E_1 and E_2.

We observe that the characteristics polynomial for the point E_1 is:

$$\lambda^3 + (\alpha+\beta)\lambda^2 + \alpha\beta\lambda + 2\beta = 0$$

Because $\beta > 0$, the system (1) does not undergo pitchfork bifurcation at the points E_1 and E_2, so we study the Hopf bifurcation at these points. The following remark characterizes the imaginary roots of (3).

Remark 2 . The condition $\beta > 0$ and the equation (3) having one real negative and two purely imaginary roots if and only if $\beta = \beta_0 = \dfrac{2-\alpha}{\alpha}$. In this case the roots of the equation (3) are: $\lambda_1 = -\dfrac{2}{\alpha}, \lambda_{2,3} = \pm iw, w = \sqrt{2-\alpha^2}$.

In the following we show that the system (1) undergo a Hopf bifurcation at the point E_1 (for E_2 is similar). Remember that β is bifurcation parameter.

Theorem 3.

If $\beta = \beta_0 = \dfrac{2-\alpha}{\alpha}$, the equation (3) has a negative solution λ_1 together with apair of purely imaginary roots $\lambda_{2,3}$ such that $\mathrm{Re}[\lambda'_\beta(\beta_0)] \neq 0$, therefore the system (1) displays a Hopf bifurcation at the point E_1.

Proof: If $\beta = \beta_0 = \dfrac{2-\alpha}{\alpha}$, then the equation (3) is transformed into:

$$(\lambda + \frac{2}{\alpha})(\lambda^2 + 2 - \alpha^2) = 0$$

With solutions $\lambda_1 = -\dfrac{2}{\alpha}$, $\lambda_2 = -i\sqrt{2-\alpha^2}$, $\lambda_3 = i\sqrt{2-\alpha^2}$.

Differentiating both sides of equation (3) with respect to β, we obtain

$$3\lambda^2 \frac{d\lambda}{d\beta} + 2\alpha\lambda\frac{d\lambda}{d\beta} + \lambda^2 + 2\beta\lambda\frac{d\lambda}{d\beta} + \lambda^2 + \alpha\beta\frac{d\lambda}{d\beta} + \alpha\lambda + 2 = 0$$

$$\frac{d\lambda}{d\beta} = -\frac{\lambda^2 + \alpha\lambda + 2}{3\lambda^2 + 2\alpha\lambda + 2\beta\lambda + \alpha\beta}$$

by setting $\beta = \beta_0 = \dfrac{2-\alpha}{\alpha}$ and $\lambda = i\sqrt{2-\alpha^2}$, we get

$$\lambda'_\beta(\beta_0) = -\frac{\alpha^2 + i\alpha\sqrt{2-\alpha^2}}{-4 + 2\alpha^2 + i\frac{4}{\alpha}\sqrt{2-\alpha^2}}$$

$$= -\frac{[-4\alpha^2 + 2\alpha^4 + 2\alpha - \alpha^3] + i[-\alpha^2 + 4\alpha - 2\alpha^3]\sqrt{2-\alpha^2}}{(-4+2\alpha^2)^2 + \frac{16}{\alpha^2}(2-\alpha^2)}$$

$$\mathrm{Re}[\lambda'_\beta(\beta_0)] = \frac{4\alpha^2 - 2\alpha^4 - 2\alpha + \alpha^3}{(-4+2\alpha^2)^2 + \frac{16}{\alpha^2}(2-\alpha^2)} \neq 0, \text{ for all } \alpha \neq 0.$$

Consequently, the system (2) display a Hopf bifurcation at $(0,0,0)$, so the system (1) display a Hopf bifurcation at the point E_1.

In the following we show that the system (1) is an almost linear system.

Definition 2. [6]

Let V be a subset of R^3 that contains the origin in its interior. Assume that F, G and H are real valued functions on V that vanish at the origin and whose partial derivatives are continuous and also vanish at the origin. Then the system

$$\dot{x} = a_1x + b_1y + c_1z + F(x,y,z)$$
$$\dot{y} = a_2x + b_2y + c_2z + G(x,y,z)$$
$$\dot{z} = a_3x + b_3y + c_3z + H(x,y,z)$$

is almost linear system.

Proposition 3. the system (1) is almost linear system at the point $O(0,0,0)$.

Proof: we write the system (1) as follows:

$$\dot{x} = y + F(x,y,z)$$
$$\dot{y} = x - \alpha y + G(x,y,z)$$
$$\dot{z} = -\beta z + H(x,y,z)$$

where $F(x,y,z) = 0, G(x,y,z) = -xz$ and $H(x,y,z) = x^2$. Then $F(0,0,0) = G(0,0,0) = H(0,0,0) = 0$, and all first partials of F, G and H continuous and vanish at the point $O(0,0,0)$. Therefore the system (1) is almost linear system at the point $O(0,0,0)$.

4. Numerical Example

Next, we shall give a numerical example of system (1). Let $\alpha = 5$, we can compute the Hopf bifurcation value $\beta_0 = -\frac{3}{5}$. The equilibrium E_1 is stable when $\beta = 6 > \beta_0$ and unstable when $\beta = -2 < \beta_0$, as shown in Figs.1 and 2 respectively.

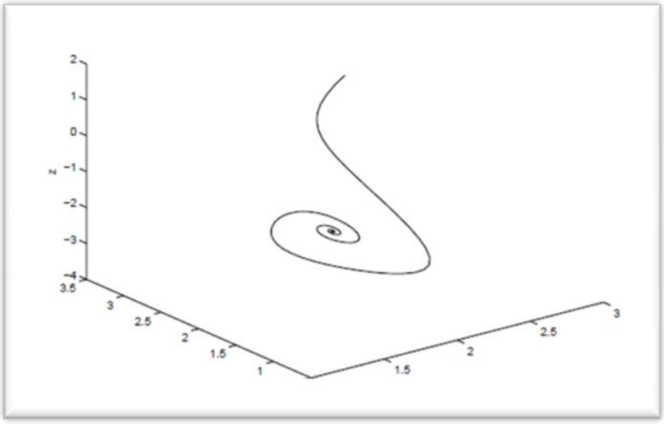

Fig. (1) Phase diagram of system (1) with $\alpha = 5,\ \beta = 6$.

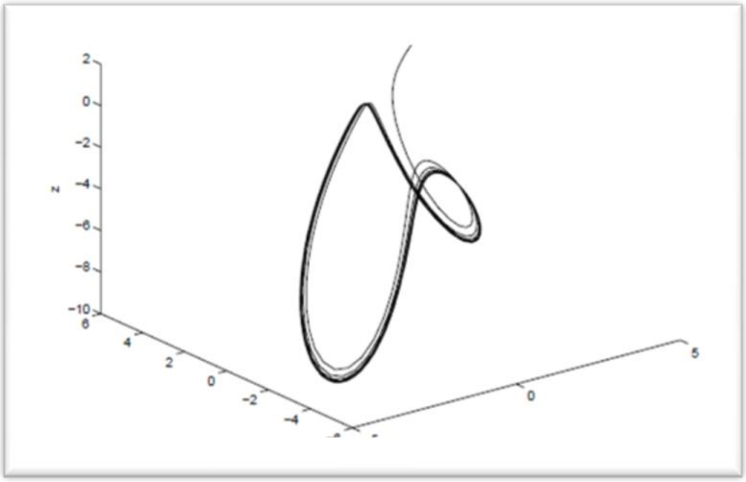

Fig. (2) Phase diagram of system (1) with $\alpha = 5, \beta = -2$

5. Conclusions

In this paper we further investigated a nonlinear differential system with three equilibrium points, origin and another two. In the origin, the system displays a pitchfork bifurcation and in the other two equilibrium points a Hopf bifurcation. Surely, there is still a lot of work, and this paper is a step in analyzing this system. the system is almost linear system at the point $O(0,0,0)$.

References

[1] Q. H. Alsafasfeh, M. S. Al-Arni, A New Chaotic Behavior from Lorenz and Rossler Systemsand Its Electronic Circuit Implementation,Circuits and Systems, Vol. 2, No. 2, (2011), pp.101-105.

[2] G. Chen and X. Dong, From Chaos to Order: Methodologies, Perspectives and Applications, World Scientific Publishing, Singapore, (1998).

[3] K. M. Cuomo, A. V. Oppenheim and S. H. Strogatz, Synchronization of Lorenz-Based Chaotic Circuits with Applications to Communications, IEEE Transactions on Circuits and Systems-II, Analog and Digital Signal Processing, Vol. 40, No. 10, (1993), pp. 626-633.

[4] E. N. Lorenz, “Deterministic Nonperiodic Flow,” Journalof Atmospheric Sciences, Vol. 20, No. 2, (1963), pp.130-141.

[5] J. Meiss, Differential Dynamical System, The Society for Industrial and Applied Mathematics, (2007).

[6] D. Gulick, Encounters with chaos, McGraw-Hill. Inc. (1992).

[7] H. Weiss, V. Weiss, The golden mean as clock cycle of brain waves, Chaos, Solitons and fractals, Vol.18, (2003), pp. 12-19.

[8] J. Ertl, E. Schafer, Brain response correlate of psychometric intelligence, Nature, (1969).

[9] J. Lu and G. Chen, A new chaotic attractor coined, Int. Journal of Bifurcation and Chaos, Vol. 12, No. 3, (2002), pp.659-661.

[10] J. Lu, G. Chen, S. Zhang, Dynamical analysis of a new chaotic attractor, Int. Journal of Bifurcation and Chaos, Vol. 12, No. 5, (2002), pp.1001-1015.

[11] C. Sparrow. The Lorenz equation: bifurcations, chaos and strange attractors, Springer-Verlag, New York, (1982).

[12] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag. New York, (1990).

[13] T. Ueta and G. Chen, Bifurcation analysis of Chen's equation, Int. Journal of Bifurcation and Chaos, Vol. 10, (2000), pp.1917-1931.

[14] T. Li, G. Chen, Y. Tang. On stability and bifurcation of Chen system, Chaos, Solitons and fractals, Vol. 19, (2004), pp. 233-250.

[15] H. K. Jassim, On Local Bifurcation and Chaos of a Three-Dimensional Nonlinear System,Journal of college of Education for Pure Science, Vol.3, No. 2 (2013), pp. 150–158.

[16] Y. Yu and S. Zhang, Hopf bifurcation in the Lu system, Chaos, Solitons and Fractals, Vol. 17, (2003), pp. 901-906.

[17] H. K. Jassim, A. E. Hashoosh, N. J. Hassan, Some dynamical properties for new system derived from Chen and T systems, Journal of Advance in Mathematical Science, Vol. 1, No. 1, (2015), pp. 70-78.

[18] I. M. Talb , H. K. Jassim, The chaotic behavior for some types of local bifurcations, Journal of Advance in Mathematical Science, Vol. 2, No. 1, (2015) pp. 115-119.